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The hamiltonian index of a 2-connected graph

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Abstract

Let G be a graph. Then the hamiltonian index $h(G)$ of G is the smallest number of iterations of line graph operator that yield a hamiltonian graph. In this paper we show that $h(G) \leq \max\{1, \frac{|V(G)|-\Delta(G)}{3}\}$ for every 2-connected simple graph G that is not isomorphic to the graph obtained from a dipole with three parallel edges by replacing every edge by a path of length $l \geq 3$. We also show that $\max\{h(G), h(\overline{G})\} \leq \frac{|V(G)|-3}{6}$ for any two 2-connected nonhamiltonian graphs G and $\overline{G}$ with at least 74 vertices. The upper bounds are all sharp.

Keywords: Hamiltonian index; Maximum degree; Complement graph; Connectivity

1. Introduction

We use [1] for terminology and notation not defined here and consider only simple finite graphs. Throughout the paper we use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of a graph G, respectively.

The *line graph* of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges share a common endvertex in G. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$. The *hamiltonian index* of a graph G, denoted by $h(G)$, is the smallest integer m such that $L^m(G)$ contains a hamiltonian cycle.

Chartrand [3] showed that the hamiltonian index of G always exists for a connected graph G that is not a path. There have already appeared many results on $h(G)$ in the literature (see [2,4,7–9,11,13,14]). A formula for determining $h(G)$ given in [12] shows that we only need to consider blocks and nontrivial paths whose edges are cut-edges when we want to determine the hamiltonian index of a graph. Note that every block is 2-connected. This motivates us to consider the hamiltonian index of 2-connected graphs.

Saražin gave an upper bound on $h(G)$ as follows.

Theorem 1 (*Saražin [8]*). *Let G be a connected simple graph on n vertices that is not a path. Then* $h(G) \leq n - \Delta(G)$.

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For 2-connected simple graphs, we improve Theorem 1 as follows.

Theorem 2. *Let G be a 2-connected simple graph of order n. If* $\Delta(G) \le n-3$, *then*

$$h(G) \le \frac{n-\Delta(G)}{3}$$

unless G is isomorphic to the graph G_l *obtained from a dipole with three parallel edges by replacing every edge by a path of length* $l = \frac{|V(G)|+1}{3} \ge 3$.

Xiong gave a relation of the hamiltonian index of a graph G and its complement graph $\overline{G}$.

Theorem 3 (*Xiong [11]*). *Let G and its complement* $\overline{G}$ *be connected graphs of order* $n \ge 61$ *that are not paths. Then either* $L(G)$ *or* $L(\overline{G})$ *is hamiltonian, and if neither G nor* $\overline{G}$ *is hamiltonian, then*

$$\max\{h(G), h(\overline{G})\} \le \frac{n-1}{2}$$

and the above equality holds if and only if either G or $\overline{G}$ *is isomorphic to the graph of order* $n = 2t-1$ *obtained by identifying one endvertex of a path of length* $t-1$ *with exactly one vertex of a complete graph of order* t.

In this paper we consider the same problem for 2-connected graphs. We improve the above theorem as follows.

Theorem 4. *Let G and* $\overline{G}$ *be 2-connected simple graphs of order* $n \ge 74$. *If neither G nor* $\overline{G}$ *is hamiltonian, then*

$$\max\{h(G), h(\overline{G})\} \le \frac{n-3}{6}.$$

In Section 2, we will give some auxiliary results which are applied in Sections 3 and 4 to prove our main results. The sharpness of Theorems 2 and 4 is presented in the last section.

2. Preliminaries

Let G be a graph. A subgraph of G is called *eulerian* if it is connected and every vertex has even degree. For any two subgraphs H_1 and H_2 of G, define the *distance* $d_G(H_1, H_2)$ between H_1 and H_2 to be the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. If $d_G(e, H) = 0$ for an edge e of G then we say that H *dominates* e. A subgraph H of G is called *dominating* if it dominates all edges of G. There is a characterization of graphs G with $h(G) \le 1$ which involves the existence of a dominating eulerian subgraph in G.

Theorem 5 (*Harary and Nash-Williams [6]*). *Let G be a graph with at least three edges. Then* $h(G) \le 1$ *if and only if G has a dominating eulerian subgraph.*

A graph is called *trivial* if it has only one vertex and is called *even* if every vertex has even degree. For a nonnegative integer k, we define $V_k(G)$ by $V_k(G) = \{x \in V(G) : d_G(x) = k\}$, where $d_G(x)$ is the degree of x in G. A *branch* in G is a nontrivial path with endvertices that do not lie in $V_2(G)$ and with inner vertices of degree two (if existing). We denote by $\mathcal{B}(G)$ the set of branches of G and by $\mathcal{B}_1(G)$ the subset of $\mathcal{B}(G)$ in which at least one endvertex has degree one. For any subgraph H of G, denote by $\mathcal{B}_H(G)$ the set of branches of G whose edges are all in H. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. In this paper, we also consider the subgraph induced by a set of edges. For $F \subseteq E(G)$, the subgraph H defined by $V(H) = V(F)$ and $E(G) = F$ is said to be the subgraph induced by F, and is denoted by $G[F]$. When we simply say an "induced subgraph", it means the subgraph induced by a set of vertices.

The following theorem can be considered as an analogue of Theorem 5 for the k-iterated line graph $L^k(G)$ of a graph G.

Theorem 6 (*Xiong and Liu [12]*). *Let G be a connected graph that is not a path and let $k \geq 2$ be an integer. Then $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$ where $EU_k(G)$ denotes the set of those subgraphs H of G which satisfy the following five conditions:*

(I) *H is an even graph;*
(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;
(III) *$d_G(H_1, H - H_1) \leq k - 1$ for every induced subgraph H_1 of H with $\emptyset \neq V(H_1) \subsetneq V(H)$;*
(IV) *$|E(B)| \leq k + 1$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$;*
(V) *$|E(B)| \leq k$ for every branch B in $\mathcal{B}_1(G)$.*

Note that if we only consider 2-connected graphs then Condition (V) in the definition of $EU_k(G)$ is superfluous.

For any subset S of $\mathcal{B}(G)$, we denote by $G - S$ the subgraph obtained from G by deleting all edges and internal vertices of branches of S. A subset S of $\mathcal{B}(G)$ is called a *branch cut* if $G - S$ has more components than G. A minimal branch cut is called a *branch-bond*. Obviously, for a connected graph G, a subset S of $\mathcal{B}(G)$ is a branch-bond if and only if $G - S$ has exactly two components. We denote by $\mathcal{BB}(G)$ the set of branch-bonds of G. A branch-bond is called *odd* if it consists of an odd number of branches. The *length* of a branch-bond $S \in \mathcal{BB}(G)$, denoted by $l(S)$, is the length of a shortest branch in S. Let $\mathcal{BB}_1(G) = \mathcal{B}_1(G)$, $\mathcal{BB}_2(G) = \{S \in \mathcal{BB}(G) \setminus \mathcal{BB}_1(G) : |S| = 1\}$, and let $\mathcal{BB}_3(G) = \{S \in \mathcal{BB}(G)\colon |S| \geq 3 \text{ and } S \text{ is odd}\}$. For $i \in \{1, 2, 3\}$, define

$$h_i(G) = \begin{cases} \max\{l(S) : S \in \mathcal{BB}_i(G)\} & \text{if } \mathcal{BB}_i(G) \neq \emptyset, \\ 0 & \text{if } \mathcal{BB}_i(G) = \emptyset. \end{cases}$$

The following result is known.

Theorem 7 (*Xiong, Broersma, Li and Li, [13]*). *Let G be a connected graph. Then*

$$h(G) \leq \max\{h_1(G), h_2(G) - 1, h_3(G) + 1\},$$

and if $h(G) \geq 1$, then

$$h(G) \geq \max\{h_1(G), h_2(G) - 1, h_3(G) - 1\}.$$

Since $h_i(G) = 0 (i = 1, 2)$ for a 2-connected graph G and $h_3(G) \leq 1$ if G is hamiltonian, Theorem 7 also holds for the graph G with $h(G) = 0$ and one can obtain the following result from the above result.

Theorem 8. *Let G be a 2-connected graph that is not a path. Then*

$$h_3(G) - 1 \leq h(G) \leq h_3(G) + 1.$$

The following characterization of eulerian graphs involves branch-bonds.

Theorem 9 (*Xiong, Broersma, Li and Li, [13]*). *A connected graph is eulerian if and only if each branch-bond contains an even number of branches.*

3. Proof of Theorem 2

In this section we present the proof of Theorem 2. By $S \triangle T$ we denote the symmetric difference $(S \setminus T) \cup (T \setminus S)$. Note that if H is an even graph, then $G[E(H) \triangle E(C)]$ is also an even graph, but $G[E(H) \triangle E(C)]$ may have more components than H.

The following well-know result will be used in our proof.

Theorem 10 (*Veldman, [10]; Xiong, [11]*). *Let G be a graph with diameter at most two. Then $L(G)$ is hamiltonian.*

Now we present the proof of Theorem 2.

Proof of Theorem 2. Suppose that G is not isomorphic to G_l for any $l \geq 3$. Throughout the proof, we let u be a vertex with maximum degree in G. If $d(u) = \Delta(G) = 2$, then Theorem 2 holds trivially. Now assume that $d(u) \geq 3$. If there is no odd branch-bond in G, i.e., $h_3(G) = 0$, then $h(G) \leq 1$ by Theorem 8, we are done. So suppose that there is at least an odd branch-bond in G. Let $\mathcal{B}_0$ be an odd branch-bond with $h_3(G) = \min\{|E(B)| : B \in \mathcal{B}_0\}$, and let $I(\mathcal{B}_0)$ be the set of inner vertices of branches in $\mathcal{B}_0$. Then we have the following fact.

Claim 1. There are at least $3(h_3(G) - 1)$ vertices in $I(\mathcal{B}_0)$ such that there are at most two vertices which are adjacent to the neighbors of u among the $3(h_3(G) - 1)$ inner vertices.

Proof of Claim 1. It follows from the fact that G is a 2-connected graph that is not isomorphic to G_l.

Claim 2. $h_3(G) \leq \frac{n-\Delta(G)}{3} + 1$.

Proof of Claim 2. By the definition of an odd branch-bond, besides u there is at least one that does not belong to $I(\mathcal{B}_0) \cup N(u)$. Hence $3(h_3(G) - 1) + (\Delta(G) - 2) + 2 \leq n$ by Claim 1, which completes the proof of Claim 2.

To prove that $h(G) \leq \frac{n-\Delta(G)}{3}$, we distinguish the following three cases.

Case 1. $3 \leq n - \Delta(G) \leq 5$.

We claim that $h(G) \leq 1$, i.e., $L(G)$ is hamiltonian. By $n - \Delta(G) \leq 5$, $|V(G) \setminus (N(u) \cup \{u\})| \leq 4$. Hence the hypothesis that G is 2-connected implies that $d_G(u, x) \leq 3$ for any vertex $x \in V(G)$ and there exist at most two vertices for which the equality is achieved. If $V(G) \setminus (N(u) \cup \{u\}) = \emptyset$, then we are done by Theorem 10. Hence we only need to consider the case that $V(G) \setminus (N(u) \cup \{u\}) \neq \emptyset$.

If there is a vertex x with $d_G(u, x) = 3$, then there is a cycle containing u and x since G is 2-connected. Now we choose a cycle C_1 containing u and x, such that

(1) C_1 contains a maximum number of vertices of $V(G) \setminus N(u)$;
(2) subject to (1), C_1 contains a maximum number of vertices of degree two.

Note that C_1 contains at least three vertices in $V(G) \setminus (N(u) \cup \{u\})$. Hence $V(G) \setminus (N(u) \cup V(C_1))$ has at most one element. Now let H be an eulerian subgraph containing $V(C_1)$ with a maximum number of edges. Then we claim that H is a dominating eulerian subgraph of G. We will prove this by contradiction. If possible, suppose that there is an edge st such that $s, t \notin V(H)$. First suppose that $s, t \in N(u)$. Then $G[\{E(H) \cup \{us, ut, st\}\}]$ is an eulerian subgraph containing $V(C_1)$ which has more edges than H, a contradiction. Now suppose that exactly one of s, t is not in $N(H)$, say $t \notin N(u)$. Then either s or t is adjacent to a vertex in $N(u) \setminus \{s\}$ since G is a 2-connected graph that is not isomorphic to G_3, say t is adjacent to $v \in N(u) \setminus \{s\}$. Then $G[E(H)\Delta E(stvus)]$ is an eulerian subgraph containing $V(C_1)$ which has more edges than H, a contradiction. This settles our claim which implies that $L(G)$ is hamiltonian by Theorem 5.

It remains to consider the case that $d_G(u, x) = 2$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$. Note that there are at most four such vertices by the hypothesis that G is a 2-connected graph with $n - \Delta(G) \leq 5$.

Let H be an eulerian subgraph containing u, such that

(1) H contains a maximum number of vertices of G;
(2) subject to (1), H contains a maximum number of edges of G.

Then we claim that H is a dominating eulerian subgraph of G, which implies that $L(G)$ is hamiltonian by Theorem 5. We will prove this by contradiction. If possible, suppose that there is an edge $st \in E(G)$ such that $\{s, t\} \cap V(H) = \emptyset$. First suppose that s, t are both in $N(u)$, then $G[E(H) \cup E(stus)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction. Next suppose that neither s nor t is in $N(u)$. If there are two distinct vertices in $N(u)$, say s' and t', which are adjacent to s and t, respectively, then $s'u, t'u \in E(H)$, and either $d_H(u) = 2$ or $d_H(u) \geq 4$ and $G[E(H) \triangle E(stt'us's)]$ is disconnected (otherwise $H' = G[E(H) \triangle E(stt'us's)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction). Hence since G is 2-connected, there is a path P of G between $G[\{s, t, s', t'\}]$ and $G[N(u) \setminus \{s', t'\}]$ such that P does not contain u. Without loss of generality, we assume that $P' = P(t', v)$ is a shortest path of G between $G[\{s, t, s', t'\}]$ and $G[N(u) \setminus \{s', t'\}]$ such that $u \notin V(P')$, with endvertices t' and $v \in N(u) \setminus \{s', t'\}$. Note that if w_1, w_2 are two vertices in $N(u)$ with $w_1w_2 \in E(H)$, then at least one of $\{uw_1, uw_2\}$ is in $E(H)$ (otherwise $G[E(H) \triangle E(uw_1w_2u)]$ is an eulerian subgraph containing u that has more edges than H, a contradiction). Hence since there are at most two vertices in $V(G) \setminus (N(u) \cup \{u, s, t\})$, $G[E(H) \triangle (E(us'stt') \cup E(P') \cup \{vu\})]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction. So s, t are adjacent to the same vertex w in $N(u)$. Hence $w \notin V(H)$ since otherwise we can add the triangle stw to H to obtain a new eulerian subgraph containing u that has more vertices than H, a contradiction. Since G is 2-connected, there is a vertex w' of $V(G) \setminus (N(u) \cup \{u\})$ that is adjacent to one of $\{s, t\}$, say, $w' \in N(t)$. By the hypothesis that $d_G(u, x) = 2$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$, w' is adjacent to $N(u)$, say, $w'w'' \in E(G)$ and $w'' \in N(u)$. Note that there is at most one vertex in $V(G) \setminus (N(u) \cup \{u, s, t, w'\})$. Hence $H' = G[E(G) \triangle E(uwstw'w''u)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction. Finally suppose that exactly one of $\{s, t\}$ is in $N(u)$, say $s \in N(u)$. If t is adjacent to a vertex $t_1 \in N(u)$, then $G[E(H) \triangle E(stt_1us)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction. Hence t is adjacent to a vertex $t_1 \notin N(u) \cup \{u\}$. By the previous arguments, we may assume $t_1 \in V(H)$. Then $G[E(H) \triangle E(ustt_1t_2u)]$ is not connected for any vertex $t_2 \in N(u) \cap N(t_1)$ (otherwise $G[E(H) \triangle E(ustt_1t_2u)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction). Hence for each $t' \in N(t_1) \cap (N(u) \setminus \{s\})$, we have that $t_1t', t'u \in E(H)$, $d_H(t') \geq 4$ and neither of $\{t_1t', t'u\}$ is in any triangle of H. Note that there are at most two vertices in $V(G) \setminus (N(u) \cup \{u, t, t_1\})$. Hence since $d_H(t_1)$ is a positive even integer, there is a vertex $t_2 \in N_H(t_1) \cap N_H(u)$ such that $N(t_2) \cap N(u) \neq \emptyset$, say $t_3 \in N(t_2) \cap N(u) \neq \emptyset$. Since t_2u is not in any triangle of H, there is at least one of $\{t_2t_3, t_3u\}$ that is not in $E(H)$. Hence $G[E(H) \triangle E(ustt_1t_2t_3u)]$ is an eulerian subgraph containing u that has more vertices than H, a contradiction. This settles the case.

Case 2. $6 \leq n - \Delta(G) \leq 8$.

We claim that $h(G) \leq 2$. By the hypothesis that $6 \leq n - \Delta(G) \leq 8$ and G is 2-connected, $d_G(u, x) \leq 5$ for any vertex $x \in V(G) \setminus N(u)$ and there is at most one vertex for which the equality is achieved. If $V(G) \setminus (N(u) \cup \{u\}) = \emptyset$, then we are done by Theorem 10. Hence we distinguish the following three subcases.

Subcase 2.1. There is a vertex x with $d_G(u, x) = 5$. Then there is a cycle C containing u and x since G is 2-connected. Note that C contains exactly 10 vertices. Hence C is a cycle of G containing all vertices of $V(G) \setminus N(u)$. Let C' be an eulerian subgraph of $G - E(C)$ with a maximum number of edges. Then $C \cup C'$ is a dominating eulerian subgraph of G, which implies that $h(G) \leq 1$ by Theorem 5.

Subcase 2.2. $d_G(u, x) \leq 4$ for any vertex $x \in V(G)$ and there is at least a vertex x_0 with $d_G(u, x_0) = 4$. Since G is 2-connected, there exists a cycle containing u and x_0. We choose a cycle C_1 containing u and x_0, such that

(1) C_1 contains a maximum number of branches of length four;
(2) subject to (1), C_1 contains a maximum number of vertices in $V(G)$.

Let C_2 be an eulerian subgraph containing $V(C_1)$ with a maximum number of edges of G and H the graph obtained from C_2 by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in H. Then we claim that $H \in EU_2(G)$, which implies that $h(G) \leq 2$ by Theorem 6. It suffices to prove that H satisfies the conditions (I)–(IV) since G is 2-connected. By the choice of H, H satisfies (I) and (II). Note that there are at most two vertices in $V(G) \setminus (V(C_1) \cup N(u))$. Hence since any vertex w in $V(H) \setminus V(C_2)$ has degree at least three, if w has no neighbor in $V(C_1)$, then w has at least two neighbors w_1, w_2 in $N(u)$ and w_1, w_2 are both in $V(C_2)$ (otherwise $C_2' = G[E(C_2) \triangle E(ww_1uw_2w)]$ is an eulerian subgraph containing $V(C_1)$ that has more edges than C_2, a contradiction). This implies that H satisfies (III). It remains to prove that H satisfies (IV). We will prove this by contradiction. If possible, suppose that there is a branch B_0 of length at least four such that $E(B_0) \cap E(H) = \emptyset$. Since $|V(G) \setminus (V(C_1) \cup N(u))| \leq 2$, B_0 has length four and u is an endvertex of B_0. Let u' be the other endvertex of B_0. Then $u' \in N(u) \cup V(C_1)$. First suppose that u' is in $N(u) \setminus V(C_1)$, then $G[E(C_2) \triangle (E(B_0) \cup \{uu'\})]$ is an eulerian subgraph containing $V(C_1)$ that has more edges than C_2, a contradiction. Next suppose that u' is in $V(C_1) \setminus \{x_0\}$, then $G[E(C_1) \triangle (E(B_0) \cup E(P(u, w)))]$, where $P(u, w)$ is the section of C_1 from u to w that does not contain x_0, is a cycle that has more vertices than C_1, which contradicts (2). Finally suppose that $u' = x_0$. Hence since G is not isomorphic to G_4, C_1 contains at most one branch of G with length four. Then $G[E(C_1) \triangle (E(B_0) \cup E(P(u, x_0)))]$, where $P(u, x_0)$ is the section of C_1 from u to x_0 that contains no branch of length four, is a cycle containing more branches of length four than C_1, this contradicts (1). This proves that H satisfies (IV). Hence $H \in EU_2(G)$ implies that $h(G) \leq 2$. This settles the subcase.

Subcase 2.3. $2 \leq d_G(u, x) \leq 3$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$. Since the empty subgraph of G with the vertex set $\bigcup_{i \geq 3} V_i(G)$ satisfies (I) and (II), we can choose a subgraph H of G with (I) and (II), such that

(1) H contains a maximum number of branches of length at least four;
(2) subject to (1), $\max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H - H_1)$ is minimized;

(3) subject to the above, H contains a minimum number of induced subgraphs F for which $d_G(F, H-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H-H_1)$.

Then we claim that $H \in EU_2(G)$. We will prove that H satisfies (IV) by contradiction. If possible, suppose that B_0 is a branch of length at least four with endvertices w_1 and w_2, such that $E(B_0) \cap E(H) = \emptyset$. Let $P(u, w_1)$ and $P(u, w_2)$ be two shortest paths from u to w_1 and w_2, respectively. Then $|E(P(u, w_i))| \leq 3$. Let H' be the subgraph obtained from $H_1 = G[E(H) \triangle (E(B_0) \cup E(P(u, w_1)) \cup E(P(u, w_2)))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in H'. Then H' is a subgraph with (I) and (II) that contains more branches of length at least four than H, a contradiction which implies that H satisfies (IV).

It remains to prove that $d_G(H_1, H-H_1) \leq 1$ for every induced subgraph H_1 of H with $\emptyset \neq V(H_1) \subsetneq V(H)$. We will prove this by contradiction. If possible, suppose that there is an induced subgraph F of H with $\emptyset \neq V(F) \subsetneq V(H)$, such that $d_G(F, H-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H-H_1) \geq 2$, then every shortest path between F and $H-F$ is a branch of G, and there are at least two branches between F and $H-F$ since G is 2-connected. Without loss of generality, suppose $u \in V(F)$. First suppose that $H-F$ is nontrivial. Take any two branches B_1, B_2 between F and $H-F$. Let $V(B_i) \cap V(F) = \{u_i\}$ and $V(B_i) \cap V(H-F) = \{v_i\}$. Then $d_G(u, u_i)$ ($i \in \{1, 2\}$) and $d_G(v_1, v_2)$ are at most two (otherwise, since G is 2-connected, there is a vertex w such that $d_G(u, w) \geq 4$, a contradiction). Let $P_i = P(u_i, u)$ ($P_3 = P(v_1, v_2)$, respectively) be a shortest path between u_i and u for $i \in \{1, 2\}$ (between v_1 and v_2, respectively). Then $|E(P_i)| \leq 2$ for $i \in \{1, 2, 3\}$. Note that if P_i has an inner vertex of degree two which is in $V(H)$, then there is an other path of H between the endvertices of P_i. Let H' be the subgraph obtained from $G[E(H) \triangle (\bigcup_{i=1}^{2}(E(P_i) \cup E(B_i)))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in H'. Then H' is a subgraph with (I) and (II) that has less induced subgraphs F for which $d_G(F, H'-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H')} d_G(H_1, H'-H_1)$ than H, contradicting (3). Now suppose that $H-F$ is trivial, then there are at least three branches between F and $H-F$. Then there are two branches B_1, B_2 between F and $H-F$, such that, there is a path $P(u_1, u_2)$ between $u_1 \in V(B_1) \cap V(F)$ and $u_2 \in V(B_2) \cap V(F)$ such that $|E(B) \cap E(P(u_1, u_2))| \leq 2$ for any branch $B \in \mathcal{B}(G)$ (otherwise, since G is 2-connected, there is a vertex w such that $d_G(u, w) \geq 4$, a contradiction). Let H' be the subgraph obtained from $G[E(H) \triangle (E(B_1) \cup E(B_2) \cup E(P(u_1, u_2)))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in H'. Then H' is a subgraph with (I) and (II) that has less induced subgraphs F for which $d_G(F, H'-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H')} d_G(H_1, H'-H_1)$ than H, contradicting (3). This implies that H satisfies (III). Hence $H \in EU_2(G)$. This settles the subcase.

Case 3. $n - \Delta(G) \geq 9$.

If $h_3(G) \leq \frac{n-\Delta(G)}{3} - 1$, then $h(G) \leq h_3(G) + 1 \leq \frac{n-\Delta(G)}{3}$ by Theorem 8 and we are done. It remains to consider the case that $\frac{n-\Delta(G)}{3} - 1 < h_3(G) \leq \frac{n-\Delta(G)}{3} + 1$ by Claim 2. Hence, $h_3(G) \geq 3$. We will prove that $h(G) \leq h_3(G) - 1$. Let H be a subgraph of G with (I) and (II) such that

(1) H contains a maximum number of branches of length at least $h_3(G) + 1$;
(2) subject to (1), $\max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H-H_1)$ is minimized;
(3) subject to the above, H contains a minimum number of induced subgraphs F for which $d_G(F, H-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H-H_1)$.

We claim that $H \in EU_{h_3(G)-1}(G)$. It suffices to prove that H satisfies the conditions (III) and (IV). We have the following fact.

Claim 3. There are at most six vertices in $V(G) \setminus (I(\mathcal{B}_0) \cup N(u) \cup \{u\})$.

Proof of Claim 3. We prove Claim 3 by contradiction. If possible, suppose that there are at least seven vertices in $V(G) \setminus (I(\mathcal{B}_0) \cup N(u) \cup \{u\})$, then $3(h_3(G) - 1) + (\Delta(G) - 2) + 1 + 7 \leq n$ by Claim 1, i.e., $h_3(G) \leq \frac{n-\Delta(G)}{3} - 1$, a contradiction. This completes the proof of Claim 3.

We will prove that H satisfies (III) by contradiction. If possible, suppose that there is an induced subgraph F of H such that $d_G(F, H-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H)} d_G(H_1, H-H_1) \geq h_3(G) - 1$, and hence there is a shortest path P between F and $H-F$ such that $|E(P)| \geq h_3(G) - 1$. Then P is a branch of G and $\bigcup_{B \in \mathcal{B}_0} E(B)$ is a subset of either of $\{E(F), E(H-F)\}$. Since G is 2-connected, there are at least two branches between F and $H-F$ and hence there is a cycle containing the two branches. We choose a cycle C' containing at least one vertex of $V(G) \setminus (V(F) \cup V(H-F))$ such that the length of a longest branch of G whose edges belong to $E(C') \cap (E(F) \cup E(H-F))$ is minimum. Then C' has at least one branch of G whose inner vertices are in $V(F) \cup V(H-F)$ with length at least

$h_3(G) - 1$(otherwise $H' = G[E(H) \triangle E(C')]$ is a subgraph with (I) and (II) that has less induced subgraphs F for which $d_G(F, H' - F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H')} d_G(H_1, H' - H_1)$ than H, a contradiction). Hence $\mathcal{B}_0$ contains exactly three branches (otherwise, since there are at least $5(h_3(G) - 1)$ inner vertices of branches in $\mathcal{B}_0$ such that there are at most four vertices which are adjacent to the neighbors of u among the $5(h_3(G) - 1)$ inner vertices, $5(h_3(G)-1) + (\Delta(G) - 5) + 2(h_3(G) - 2) + 5 \leq n$, i.e., $h_3(G) \leq \frac{n-\Delta+9}{7}$ which implies that $h_3(G) \leq \frac{n-\Delta(G)}{3} - 1$ since $h_3(G)$ is an integer, a contradiction). Note that C' contains an even number of branches in $\mathcal{B}_0$ by the definition of odd branch-bonds. First suppose that C' contains no branches of $\mathcal{B}_0$, then there are at least $2(h_3(G) - 2) + 3 + 2 \geq 7$ vertices in $V(G) \setminus (I(\mathcal{B}_0) \cup N(u) \cup \{u\})$, which contradicts Claim 3. Now suppose that C' contains exactly two branches B_1, B_2 in $\mathcal{B}_0$. Then there are at least one of B_1, B_2 with length at least $h_3(G) + 1$ (otherwise, if the inner vertices of both of B_1, B_2 belong to $V(F) \cup V(H - F)$, then by Claim 1, $3(h_3(G)-1) + (\Delta(G)-2) + 2(h_3(G)-2) + 1 + 5 \leq n$ which implies that $h_3(G) \leq \frac{n-\Delta(G)}{3} - 1$ since $h_3(G)$ is an integer, a contradiction; if there is at most one of B_1, B_2 has inner vertices in $V(F) \cup V(H - F)$, then $H' = G[E(H) \triangle E(C')]$ is a subgraph with (I) and (II) that has less induced subgraphs F for which $d_G(F, H' - F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H')} d_G(H_1, H' - H_1)$ than H, a contradiction). By the choice of C', the branch in $\mathcal{B}_0 \setminus \{B_1, B_2\}$ has length at least $h_3(G)+1$. Hence since there are at least $(h_3(G)-2)+4$ vertices in $V(G) \setminus (I(\mathcal{B}_0) \cup N(u) \cup \{u\})$, $2h_3(G) + (h_3(G) - 1) + (\Delta(G) - 2) + 1 + (h_2(G) - 2) + 4 \leq n$ by Claim 1. So $h_3(G) \leq \frac{n-\Delta(G)}{4}$, which implies that $h_3(G) \leq \frac{n-\Delta(G)}{3} - 1$ since $h_3(G)$ is an integer, a contradiction. This proves that H satisfies (III).

It remains to prove that H satisfies (IV). We will prove this by contradiction. If possible, suppose that there is a branch $B_0 \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$ with $|E(B_0)| \geq h_3(G) + 1$. Let u and v be two endvertices of B_0 and $S(u, B_0)$ be a branch-bond containing B_0 such that any branch of $S(u, B_0)$ has u as an endvertex. Since G is 2-connected, $|S(u, B_0)| \geq 2$.

By the following algorithm, we first find a cycle of G that contains B_0 and then obtain a contradiction.

Algorithm B_0.

1. If $|S(u, B_0)|$ is even, then select a branch $B_1 \in S(u, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\})$ by Theorem 9. Otherwise, since $|E(B_0)| \geq h_3(G) + 1$, select a branch $B_1 \in S(u, B_0)$ with
$$|E(B_1)| = l(S(u, B_0)) \leq h_3(G)$$
(obviously $B_1 \neq B_0$) and let $u_1 (\neq u)$ be the other endvertex of B_1. If $u_1 = v$, then set $t := 1$ and stop. Otherwise $i := 1$.
2. Select a branch-bond $S(u, u_i, B_0)$ in G which contains B_0 but not $B_1, B_2, \ldots, B_i$ such that any branch in $S(u, u_i, B_0)$ has exactly one endvertex in $\{u, u_1, u_2, \ldots, u_i\}$. If $|S(u, u_i, B_0)|$ is even, then, by Theorem 9, select a branch
$$B_{i+1} \in S(u, u_i, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\}).$$
Otherwise, since $|E(B_0)| \geq h_3(G) + 1$, select a branch $B_{i+1} \in S(u, u_i, B_0)$ such that
$$|E(B_{i+1})| = l(S(u, u_i, B_0)) \leq h_3(G)$$
(obviously $B_{i+1} \neq B_0$), and let u_{i+1} be the endvertex of B_{i+1} that is not in $\{u, u_1, u_2, \ldots, u_i\}$.
3. If $u_{i+1} = v$, then set $t := i + 1$ and stop. Otherwise replace i by $i + 1$ and return to step 2.

Note that $|\mathcal{B}(G)|$ is finite, and $d_G(v) \geq 2$ implies that Algorithm B_0 will stop after a finite number of steps. Note that $G[\bigcup_{i=0}^{t} E(B_i)]$ is connected. Furthermore, since $u_t = v$ and $|S(u, u_i, B_0)| \geq 2$, $G[\bigcup_{i=0}^{t} E(B_i)]$ has a cycle C_0 of G which contains B_0. Let H' be the subgraph of G obtained from $G[E(H) \triangle E(C_0)]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in H'. By the choice of B_i, $|E(B)| \leq h_3(G)$ for $B \in \mathcal{B}_H(G) \cap \{B_1, B_2, \ldots, B_t\}$. Hence, since B_0 is in C_0, H' is a subgraph with (I) and (II) that has less branches of length $h_3(G) + 1$ than H, a contradiction. Hence H satisfies (IV). This proves that $H \in EU_{h_3(G)-1}(G)$, which implies that $h(G) \leq h_3(G) - 1 \leq \frac{n-\Delta(G)}{3}$. The proof of Theorem 2 is completed. □

By the proof of Theorem 2, one can obtain the following result which answers partly a question proposed in [13].

Theorem 11. *Let G be a 2-connected simple graph of order n. Then if $h_3(G) \geq \frac{n-\Delta(G)-2}{3} \geq \frac{7}{3}$ then*
$$h(G) = h_3(G) - 1.$$

4. Proof of Theorem 4

Before giving the proof of Theorem 4, we state the following well-known result involving the vertex degree sequence of a graph.

Theorem 12 (*Chvátal [5]*). *Let G be a simple graph with vertex degrees $d_1 \le d_2 \le \cdots \le d_n$, where $n \ge 3$. If $i < \frac{n}{2}$ implies that $d_i > i$ or $d_{n-i} \ge n - i$, then G is hamiltonian.*

We need its following consequence. By $\lceil x \rceil$ we denote the minimum integer not less than x.

Theorem 13. *Let G be a simple graph of order $n \ge 14$ with $h_3(G) \ge \frac{n+4}{6}$. Then $\overline{G}$ is hamiltonian.*

Proof of Theorem 13. Let $\mathcal{B}_0$ be an odd branch-bond in $\mathcal{BB}_3(G)$ such that $h_3(G) = \min\{|E(B)| : B \in \mathcal{B}_0\}$. Then there are at least $3(h_3(G) - 1) \ge \frac{n}{2} - 1$ inner vertices of branches of $\mathcal{B}_0$ which have degree exactly $n - 3$ in $\overline{G}$ and there is at least one vertex w of G that is not adjacent to $\lceil \frac{n}{2} \rceil - 1$ vertices of the $3(h_3(G) - 1)$ inner vertices and the neighbor of one of the $\lceil \frac{n}{2} \rceil - 1$ vertices. Hence w has at least $\lceil \frac{n}{2} \rceil - 1 + 1$ neighbors in $\overline{G}$ and so $d_{\overline{G}}(w) \ge \frac{n}{2}$. The other $n - 3(h_3(G) - 1) - 1$ vertices have degree at least $3(h_3(G) - 2) \ge \frac{n-8}{2} \ge 3$ in $\overline{G}$. Hence $\overline{G}$ satisfies the conditions of Theorem 12, which implies that $\overline{G}$ is hamiltonian. □

Now we present the proof of Theorem 4.

Proof of Theorem 4. Without loss of generality, we can assume that neither G nor $\overline{G}$ is hamiltonian, and $L(\overline{G})$ is hamiltonian. Then $\Delta(G) \ge \frac{n-1}{2}$ since otherwise $\delta(\overline{G}) = n - 1 - \Delta(G) \ge \frac{n}{2}$ which implies that $\overline{G}$ is hamiltonian by Theorem 12, a contradiction.

We distinguish the following cases to finish our proof.

Case 1. $h_3(G) \le \frac{n-9}{6}$.

By Theorem 8, $h(G) \le h_3(G) + 1 \le \frac{n-3}{6}$ and then we are done.

Case 2. $\frac{n-9}{6} < h_3(G) < \frac{n+4}{6}$.

Let $\mathcal{B}_0$ be an odd branch-bond in $\mathcal{BB}_3(G)$ such that $h_3(G) = \min\{|E(B)| : B \in \mathcal{B}_0\}$. Then $\mathcal{B}_0$ has exactly three branches. We will prove this by contradiction. If possible, suppose that $\mathcal{B}_0$ has at least five branches, then $\mathcal{B}_0$ has at least $5(h_3(G) - 1) \ge \frac{n}{2}$ inner vertices of degree 2 in G. Hence $\overline{G}$ has at least $\lceil \frac{n}{2} \rceil$ vertices of degree $n - 3$ in $\overline{G}$ and the other vertices has degree at least $3(h_3(G) - 3) > \frac{n-27}{2} \ge 23$. Hence $\overline{G}$ satisfies the conditions of Theorem 12, which implies that $\overline{G}$ is hamiltonian, a contradiction. Hence $\mathcal{B}_0$ has exactly three branches. Let $B_0 \in \mathcal{B}_0$ with $|E(B_0)| = h_3(G)$. Taking two branches B_1, B_2 that are not B_0, there is a cycle C_0 in G containing B_1, B_2 by the definition of an odd branch-bond. Now let H be the subgraph obtained from C_0 by adding all vertices in $(\bigcup_{i=3}^{\Delta(G)} V_i(G)) \setminus V(C_0)$ as isolated vertices in H. Then H satisfies (I) and (II). We claim that $H \in EU_{h_3(G)-1}(G)$. It suffices to prove that H satisfies (III) and (IV). Since $\Delta(G) \ge \frac{n-1}{2}$ and $\frac{n-9}{6} < h_3(G) < \frac{n+4}{6}$, there are at most $n - 3(h_3(G) - 2) - \Delta(G) - 2 < 9$ vertices outside of $B_0 \cup N(u) \cup \{u\}$, where u is a vertex of G with maximum degree $\Delta(G)$. Hence $|E(B)| \le 9 \le h_3(G) - 2$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$, which implies that H satisfies (III) and (IV). Hence $H \in EU_{h_3(G)-1}(G)$, which implies that $h(G) \le h_3(G) - 1 \le \frac{n-3}{6}$ by Theorem 8.

Case 3. $h_3(G) \ge \frac{n+4}{6}$.

By Theorem 12, $\overline{G}$ is hamiltonian, a contradiction. This completes the proof of Theorem 4. □

5. Sharpness

In this section, we discuss the sharpness of Theorems 2 and 4.

First we show that the result in Theorem 2 is sharp. To see this, we construct a graph G_0 from the graph G_l ($l \ge 2$) depicted in Theorem 2 and a nontrivial complete graph $K_{\Delta-2}$ of order $\Delta - 2$. Let w be a vertex of degree three in G_l and wu an edge of G_l. Now divide wu and denote the new vertex by x, and obtain the graph G_0 by identifying w with exactly one vertex of $K_{\Delta-2}$ and adding an edge xv, where v is a vertex of $K_{\Delta-2}$ that is not the identified vertex. Note that G_0 is a 2-connected simple graph such that $\Delta(G_0) = \Delta$, $|V(G_0)| - \Delta = 3(l - 1) \ge 3$ and

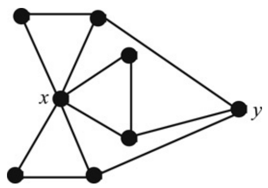

Fig. 1. The graph $G(3)$.

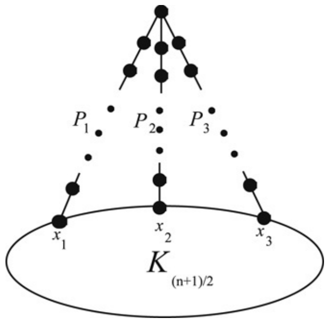

Fig. 2. The graph G_{00}.

$h_3(G_0) = l = \frac{|V(G_0)|-\Delta+3}{3}$. By Theorem 8, $h(G_0) \geq h_3(G_0) - 1 = \frac{|V(G_0)|-\Delta}{3}$. Let C be a longest cycle of G_0. Then $C \in EU_{h_3(G_0)-1}(G_0)$. Hence $h(G_0) \leq h_3(G_0) - 1 = \frac{|V(G_0)|-\Delta}{3}$ by Theorem 6. Thus $h(G_0) = \frac{|V(G_0)|-\Delta(G_0)}{3}$.

We cannot relax the condition on $\Delta(G)$ in Theorem 2. For example we consider the graph $G(k)$ obtained from $k \geq 3$ triangles $x_{i,1}x_{i,2}x_{i,3}$ $(i = 1, 2, \ldots, k)$ and a vertex y by identifying $x_{1,1}, x_{2,1}, \ldots, x_{k,1}$ and joining $x_{i,2}$ to y for each i, for $k = 3$, see Fig. 1.

Then $G(k)$ is 2-connected graph with $\Delta(G) = |V(G)| - 2 = 2k$. By x we denote the vertex obtained by identifying $x_{1,1}, x_{2,1}, \ldots, x_{k,1}$. Since we obtain $k \geq 3$ components after deleting two vertices x and y, $G(k)$ is not hamiltonian for any integer $k \geq 3$. Hence $h(G(k)) > \frac{|V(G(k))|-\Delta(G(k))}{3}$.

Note that using more discussion we can improve the bound 74 on the order of G in Theorem 4. We show the bound on $\max\{h(G), h(\overline{G})\}$ is sharp in Theorem 4 by describing the following graph. Take an integer $n \geq 9$ with $n \equiv 3 (\mathrm{mod}\ 6)$. Let P_1, P_2, P_3 be three vertex-disjoint paths of length $\frac{n+3}{6}$ and let $K_{\frac{n+1}{2}}$ be a complete graph of order $\frac{n+1}{2}$. Take three vertices x_1, x_2, x_3 of $K_{\frac{n+1}{2}}$. Now let G_{00} be the graph obtained from P_i and $K_{\frac{n+1}{2}}$ by identifying x_i and one endvertex of P_i, and identifying the other endvertices of P_1, P_2, P_3, respectively, see Fig. 2.

Obviously, $|V(G_{00})| = n$. Then G_{00} and $\overline{G}_{00}$ are both 2-connected. Since $(\bigcup_{i=1}^{3} V(P_i)) \setminus \{x_1, x_2, x_3\}$ is a cut set of $\frac{n-1}{2}$ vertices whose deletion yields $\frac{n+1}{2} = \frac{n-1}{2} + 1$ components of $\overline{G}$, $\overline{G}$ is not hamiltonian. Let C be a longest cycle of G_0. Then $C \in EU_{h_3(G_{00})-1}(G_{00})$, which implies that $h(G_{00}) \leq h_3(G_{00}) - 1$ by Theorem 6. Note that $h_3(G_{00}) = \frac{n+3}{6}$. By Theorem 8, $h(G_{00}) \geq h_3(G_{00}) - 1 = \frac{n-3}{6}$. Hence $h(G_{00}) = \frac{|V(G_{00})|-3}{6}$. The graph G_{00} shows that the upper bound on $h_3(G)$ in Theorem 13 is also sharp.

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